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Differential Migdal–Kadanoff renormalization group for disordered systems

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Abstract. Phase transitions in quenched random systems are studied within the Migdal-Kadanoff approximation. Exact differential recursion relations for the distribution of coupling are derived. The q-state Potts model with random ferromagnetic couplings is studied in two and three dimensions. In three dimensions there is a non-perturbative random fixed point which exists when the specific heat exponent is either positive or slightly negative. If the specific heat exponent is slightly negative a multicritical point separates strong and weak disorder regimes. The analytic results are confirmed by numerical simulations of the recursion relations.

1. Introduction

Phase transitions in systems with quenched random interactions have provided particularly stubborn problems in theoretical statistical physics. Real space renormalization group studies of these systems have met with mixed success. Though real space methods are inherently uncontrolled approximations they often yield important qualitative insights and can be directly applied in physical dimensions. Real space studies of random systems are rendered difficult by the fact that the renormalization group acts in a space of probability distributions of coupling strengths. This difficulty has been circumvented in several ways in past studies. Some investigators have replaced the full probability distribution by two delta functions [1, 2]. This approach is useful for studying the qualitative features of the phase diagram and the crossover from percolative to magnetic phase transitions. When a random fixed point exists, however, the two delta function approximation does not adequately describe the fixed point.

The Migdal-Kadanoff bond moving scheme is the starting point for a number of past studies of random spin systems. The Migdal-Kadanoff approach is equivalent to replacing the original Euclidean lattice by a hierarchical lattice and then carrying out an exact decimation. The ferromagnetic Potts model with quenched disorder on a hierarchical lattice is a convenient system in which to study the effects of disorder since disorder can be made relevant or irrelevant by varying the number of Potts components‡. Andelman and Berker [3] examined the distribution of couplings for this system on a hierarchical lattice resulting from applying the Migdal-Kadanoff bond

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‡ On hierarchical lattices pure Potts models always have continuous transitions in contrast to Euclidean lattices where first-order transitions occur for sufficiently large q and d.

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moving scheme in two-dimensions. In [3] they use a numerical method to study the random fixed point of the renormalization group. Derrida and Gardner [4] studied the same system analytically. They derived recursion relations for the moments of the probability distribution and found fixed points and exponents using a moment expansion of the distribution.

In the present paper we develop a more general analytic method for applying the Migdal–Kadanoff renormalization group to random system [5]. The central result is an exact differential recursion relation for the Laplace transform of the distribution of couplings. Differential as opposed to difference equations are obtained by continuing the value of the length rescaling factor, b to one. In the case of pure systems taking the limit $b \to 1$ leads to desirable properties; exact duality is preserved in two dimensions, difference equations are replaced by more convenient differential equations, dimensionality becomes a continuously adjustable parameter and the values of exponents are slightly improved. We expect these features to be inherited by the renormalization group for random systems when $b \to 1$. However, there is a possible difficulty with the $b \to 1$ limit. If b is a positive integer, the Migdal–Kadanoff approach is equivalent to exact decimation on a realizable hierarchical lattice. This means that the renormalization group is a well-defined transformation from N random variables to N/b^d random variables and we are guaranteed that the distribution of couplings exist at each stage of renormalization. When the limit $b \to 1$ is taken, the renormalization group no longer corresponds to a well-defined transformation on a set of random variables. The existence and significance of the solutions of the recursion relations are less clear. As we shall see however, interesting and plausible solutions emerge in the $b \to 1$ limit.

The general formalism is developed in section 2 for the disordered ferromagnetic Potts model. In section 3 the differential equation for the distribution is used to obtain recursion relations for the cumulants of the distribution and fixed points are found in the subspace of the first four cumulants. In two dimensions, a perturbative random fixed point is found in which the specific heat exponent, α, is the small parameter. This fixed point and the associated exponents are the $b \to 1$ continuation of the results obtained in [4] for $b = 2$.

In three dimensions we find an unexpected result. The perturbative fixed point is unphysical for $\alpha > 0$ and the critical properties are controlled by a non-perturbative fixed point. This fixed point survives for small negative values of α until it is annihilated by merging with the unstable perturbative fixed point. Section 4 contains a Monte Carlo study of the q-state Potts model on a $b = 2$ hierarchical lattice in two and three dimensions. The numerical results are in good agreement with the analytic work. The paper closes with a discussion in section 5.

2. Differential Migdal–Kadanoff renormalization group for disordered Potts models

The Hamiltonian for the q-state Potts model with random couplings is given by

$$-\mathcal{H}/k_B T = \sum_{\langle ij \rangle} K_{ij}(\delta_{S_i S_j} - 1) \tag{2.1}$$

where the spins, S_i take integer values $S_i = 1, 2, 3, \ldots, q$. The sum is over nearest-neighbour sites, $\langle ij \rangle$, on a d-dimensional lattice. The couplings, K_{ij} are taken to be non-negative, independent, identically distributed random variables. The cumulative distribution, H for K is defined by

$$H(x) = \text{Prob}\{K_{ij} < x\}. \tag{2.2}$$

The Migdal–Kadanoff bonding moving scheme is equivalent to replacing the Euclidean lattice by a hierarchical lattice. In the present work we take the necklace hierarchical lattice which has the unit cell shown in figure 1. The full lattice is constructed by iteratively replacing each bond by a unit cell until a lattice of the desired size is obtained. The unit cell consists of b groups of b^{d-1} parallel bonds connected in series. The renormalization group transformation consists of adding the b^{d-1} parallel bonds in each set and then decimating the resulting b bonds in series.

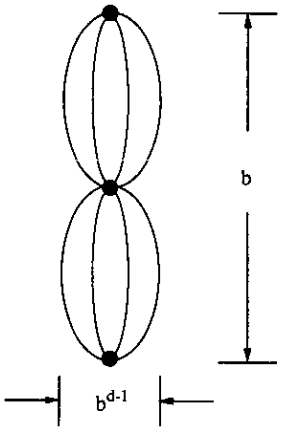

Figure 1. The unit cell for the necklace hierarchical lattice.

The transformation from the b^d original couplings, $K_1, \ldots, K_{b^d}$, to the renormalized coupling, $\tilde{K}$ is easily obtained using standard methods [2]. The key idea is that there is a change of variables, $K \to W = w(K)$ for which the decimation step corresponds to adding random variables. Decimating b random bonds, $K_1, \ldots, K_b$ in series yields an effective bond of strength $w^{-1}(w(K_1) + \ldots + w(K_b))$ where,

$$w(x) = \ln\left[\frac{e^x + q - 1}{e^x - 1}\right] \tag{2.3}$$

and w^{-1} is the functional inverse of w. It is easily verified that $w^{-1} = w$.

The full recursion relations thus take the form

$$\tilde{K} = w^{-1}\left(\sum_{j=1}^{b} w\left(\sum_{i=1}^{b^{d-1}} K_{ij}\right)\right). \tag{2.4}$$

The recursion relations (2.4) induce recursion relations for the probability distribution for K. A sum of independent random variables corresponds to the convolution of the corresponding probability densities. This fact suggests the use of the Laplace transform to convert the convolution into a product. Define the Laplace transform of a cumulative distribution H as

$$h(z) = L[H] = \int_0^\infty e^{-zx}\, dH(x). \tag{2.5}$$

Similarly, let $g(z)$ be the Laplace transform of the cumulative distribution, G, for W. Since $w(x)$ is a monotone decreasing function, the relation between G and H is

$$G(x) = 1 - H(w(x)). \tag{2.6}$$

The first step in the renormalization group transformation is a sum of bK's so that h is transformed according to

$$h(z) \to h(z)^b. \tag{2.7}$$

The next step is changing variables to W. To accomplish this, take the inverse Laplace transform, compose the result with w and then, to obtain $g(z)$, take the Laplace transform. We may formally define an operator $\boldsymbol{P}$ which accomplishes this,

$$\boldsymbol{P}[f] = -\boldsymbol{L}[\boldsymbol{L}^{-1}[f] \circ w] \tag{2.8}$$

where $\circ$ is functional composition and $\boldsymbol{L}^{-1}$ is the inverse Laplace transform for cumulative distributions such that if

$$f = \boldsymbol{L}[F] \tag{2.9}$$

then

$$F(x) = \boldsymbol{L}^{-1}[f] = \int_0^x \mathrm{d}x' \int_C \frac{\mathrm{d}z}{2\pi \mathrm{i}} \mathrm{e}^{zx'} f(z)$$

with C an appropriate Bromwich contour. The minus sign in the definition (2.8) compensates for the minus sign relating G and H in (2.6). Thus, since $w = w^{-1}$ we have

$$g = \boldsymbol{P}[h] \qquad \text{and} \qquad h = \boldsymbol{P}[g]. \tag{2.10}$$

Note that $\boldsymbol{P}$ is a linear operator transforming function of z into functions of z. Properties of $\boldsymbol{P}$ are given in the appendix.

With the operator $\boldsymbol{P}$ and the rule for adding random variables we can transcribe the recursion relations, (2.4) onto recursion relations for h,

$$\tilde{h} = \boldsymbol{P}[\boldsymbol{P}[h^b]^{b^{d-1}}]. \tag{2.11}$$

To obtain differential recursion relations set $b = 1 + \mathrm{d}l$ and keep terms to leading order in $\mathrm{d}l$,

$$\frac{\partial h}{\partial l} = (d-1)h \ln h + \boldsymbol{P}[\boldsymbol{P}[h] \ln \boldsymbol{P}[h]] \tag{2.12}$$

where the dependence on z and the length scale l is implicit. Although (2.12) no longer corresponds to a realizable hierarchical lattice, we assert that it is the natural continuation to $b \to 1$ for random systems. Alternatively, by acting with $\boldsymbol{P}$ we obtain recursion relations for $g(z)$,

$$\frac{\partial g}{\partial l} = g \ln g + (d-1)\boldsymbol{P}[\boldsymbol{P}[g] \ln \boldsymbol{P}[g]]. \tag{2.13}$$

Equation (2.13) is equivalent to (2.12) but, for a given finite truncation scheme, (2.13) may yield a different result from (2.12) as discussed in section 3.3.

We begin the study of (2.12) by deriving the differential recursion relations for pure systems. Suppose that the probability density for K is a delta function at k so that, $h(z) = \exp(-kz)$. From (A5) we have $g(z) = \boldsymbol{P}[\exp(-kz)] = \exp(-w(k)z)$ so that (2.14) becomes

$$\frac{\partial \mathrm{e}^{-kz}}{\partial l} = (d-1)(-kz)\,\mathrm{e}^{-kz} - \boldsymbol{P}[\mathrm{e}^{-w(k)z} w(k) z]. \tag{2.14}$$

Dividing out $\exp(-kz)$ and using (A12) we obtain the pure system recursion relations,

$$\frac{\partial k}{\partial l} = (d-1)k + \frac{w(k)}{w'(k)} \equiv R[k] \tag{2.15}$$

from which an equation for the fixed point, k^*, is obtained,

$$k^* = -w(k^*)/[(d-1)w'(k^*)]. \tag{2.16}$$

The correlation length exponent, ν, is the inverse of the derivative of R at k^*,

$$\frac{1}{\nu} = d - \frac{w(k^*)w''(k^*)}{w'(k^*)^2}. \tag{2.17}$$

The pure system results are in agreement with [6]. In the next section we show how to use (2.12) to obtain recursion relations for the cumulants of the distribution.

3. Recursion relations for cumulants

3.1. General form of the recursion relations

Suppose that the probability law for K remains well-behaved under renormalization so that it is characterized by its cumulants; then $h(z)$ may be written as

$$h(z) = \exp\left(-k^* z + \sum_{n=1} (-1)^n \frac{c_n z^n}{n!}\right) \tag{3.1}$$

where c_1 is the deviation of the mean value of K from the pure fixed point, $k^* + c_1 = \langle K \rangle$, and the c_n for $n > 1$ are the cumulants of the distribution of K. For example $c_2 = \langle K^2 \rangle - \langle K \rangle^2$ is the variance of K. To obtain recursion relations for the c_n, divide (2.14) by h to obtain a differential equation for $\ln h(z)$,

$$\frac{\partial \ln h}{\partial l} = (d-1)\ln h + \frac{1}{h} \boldsymbol{P}[\boldsymbol{P}[h] \ln \boldsymbol{P}[h]]. \tag{3.2}$$

The coefficient of z^n in a Taylor series expansion of the right-hand side of (3.2) determines the recursion relation for the nth cumulant. The first step in obtaining this coefficient is expanding $h(z)$ in the form,

$$h(z) = e^{-k^* z}(1 + a_1 z + a_2 z^2 + \dots) \tag{3.3}$$

where the coefficients, a_i are polynomials in the cumulants. The next step is applying the operator $\boldsymbol{P}$ to $h(z)$. Define the quantity $p(n, x)$ as

$$p(n, x) \equiv \boldsymbol{P}[z^n e^{-zw(x)}] \tag{3.4}$$

then

$$\boldsymbol{P}[h] = p(0, w^*) + a_1 p(1, w^*) + a_2 p(2, w^*) + \dots \tag{3.5}$$

$w^* = w(k^*)$. In the appendix we obtain the following recursive rule for $p(n, x)$,

$$\begin{aligned} p(0, x) &= e^{-zx} \\ p(n+1, x) &= -\frac{1}{w'(x)} \frac{\partial}{\partial x} p(n, x). \end{aligned} \tag{3.6}$$

Thus an explicit expression for $g = \boldsymbol{P}[h]$ may be obtained iteratively. The remaining steps are straightforward and tedious: $\boldsymbol{P}[h] \ln \boldsymbol{P}[h]$ is expanded in a series in $z^n \exp(-zw^*)$, equations (3.6) are applied again with $x \to k^*$, the result is divided by $h(z)$ to obtain, $(1/h)\boldsymbol{P}[\boldsymbol{P}[h] \ln \boldsymbol{P}[h]]$ which is expanded as a Taylor series in z and plugged into (3.2). Finally, the coefficient of z^n is extracted and multiplied by $(-1)^n n!$ to obtain the recursion relation for c_n. We carried out these formal manipulations using *Mathematica*.

In order to truncate the hierarchy of cumulant recursion relations in a systematic way each cumulant is multiplied by some power of a small parameter, λ and terms to a given order in λ are retained at each step in the calculation. At the end of the calculation, λ is set to one. We make the substitutions $c_1 \to \lambda c_1$, $c_2 \to \lambda c_2$, $c_3 \to \lambda^2 c_3$, $c_4 \to \lambda^3 c_4$ and let all the higher cumulants vanish. Terms of order λ^3 are kept. For general q and d the coefficients are too complicated to display; however, the structure of the recursion relations to order λ^3 is,

$$\begin{aligned}
\frac{dc_1}{dl} &= \frac{1}{\nu} c_1 + \text{all other terms of order } \lambda, \lambda^2 \text{ and } \lambda^3 \\
\frac{dc_2}{dl} &= \frac{\alpha}{\nu} c_2 + \text{all terms of order } \lambda^2 \text{ and } \lambda^3 \text{ except } c_1^2, c_1^3 \\
\frac{dc_3}{dl} &= \frac{1}{\nu}(3 - 2\,d\nu) c_3 + \text{all other terms of order } \lambda^2 \text{ and } \lambda^3 \text{ except } c_1^2, c_1^3 \\
\frac{dc_4}{dl} &= \frac{1}{\nu}(4 - 3\,d\nu) c_4 + A c_3 c_2^2 + B c_2^3
\end{aligned} \tag{3.7}$$

where ν is given in (2.17), $\alpha = 2 - d\nu$ is the specific heat exponent for the pure system and A and B are constants. The ordering scheme is self-consistent insofar as no terms appear in the equation for a given cumulant having a lower order than the given cumulant. For example, the recursion relation for c_4 contains c_2^3 but not c_2^2. Furthermore, to linear order, the equation for c_n depends only on cumulants c_m with $m \geq n$. Thus the recursion relations linearized at the pure fixed point ($c_1 = c_2 = \ldots = 0$) form a triangular matrix and the crossover exponents, ϕ_n, can be read off of the leading terms in the recursion relation, $\phi_n = n - (n-1)\,d\nu$, $n = 1, 2, 3, \ldots$.

Note that the crossover exponent, ϕ_2, associated with the second cumulant is the specific heat exponent and that the pure fixed point is stable against small perturbations when $\alpha < 0$ in agreement with the Harris criterion. This is not surprising since the Harris criterion is satisfied for all positive integer values of b [3, 7].

3.2. Perturbative fixed point for small α

Since the sign of α determines the stability of the pure fixed point it is tempting to suppose that for small positive α there will be a random fixed point whose cumulants become small as powers of α. An examination of the recursion relations for $c_1, \ldots, c_4$, equations (3.6), show that there is such a solution with $c_1 \sim c_2 \sim \alpha$, $c_3 \sim \alpha^2$ and $c_4 \sim \alpha^3$. Supposing that the higher cumulants are higher order in α we can study this perturbative random fixed point to order α^2. To this order, all of the terms not explicitly displayed in (3.6) can be evaluated at the value of $q = q_c(d)$ for which α vanishes; $q_c(2) = 6.314\,48$ and $q_c(3) = 17.1413$. To order α^2, $c_4 = 0$, and the equation for c_3 can be solved in terms of c_2. This solution is plugged into the equation for c_2 which is then solved for c_2 in

terms of c_1. This solution is plugged into the equation for c_1 and this equation is solved for c_1 to linear order in α. The results for $d=2$ and $d=3$ are shown in table 1. For comparison, the analogous perturbative fixed point found by Derrida and Gardner [4] for $d=2$ and $b=2$ is also shown in table 1. Linearizing the recursion relations around the perturbative fixed point yields a matrix of coefficients whose eigenvalues determine the critical exponents. Having kept terms of order α^2 we are able to calculate the eigenvalues of the renormalization group to order α. The two leading eigenvalues determine ν, the correlation length exponent, and ϕ, the leading crossover exponent. These are given in table 1 along with the results obtained in [4].

For $\alpha > 0$, the two-dimensional perturbative random fixed point corresponds to a well-defined probability distribution, the correlation length exponent satisfies the bound [8] $\nu > 2/d$ and the crossover exponent is minus the pure specific heat exponent so that the fixed point is stable. Not surprisingly, the $b \to 1$ results are quite close to the $b=2$ results of [4]. We refer the reader to [4] for a more detailed discussion of the perturbative fixed point.

3.3. *Non-perturbative fixed point for $d=3$*

In three dimensions the perturbative fixed point is unphysical for $\alpha > 0$ since the fixed 'distribution' has a negative variance. Thus we are lead to search for a non-perturbative random fixed point. In principle such a fixed point need not have small or even finite moments; however, as it turns out, the cumulants are reasonably small for $d=3$ and can be accurately deduced from a few terms of the cumulant recursion relations. We searched numerically for fixed points of the recursion relations (3.7) for $d=3$ and several values of q near $q_c(3)$.

For $q = 17.2343 > q_c(3)$ the pure specific heat exponent is small and positive ($\alpha = 0.002$). There are two physical solutions to the recursion relations, the pure fixed point and a non-perturbative fixed point. The cumulants and exponents of the non-perturbative fixed point are given in table 2. Note that this fixed point is stable and that the

Table 1. The perturbative fixed point in two and three dimensions. The middle row is the result obtained from [4] for the case $b=2$.

	c_1	c_2	c_3	ν	ϕ
$d=2$	0.2912α	1.464α	$2.557\alpha^2$	$1+0.1159\alpha$	$-\alpha$
$d=2, b=2$ [4]	0.4966α	2.111α	$3.918\alpha^2$	$1+0.1306\alpha$	$-\alpha$
$d=3$	1.380α	-5.191α	$93.63\alpha^2$	$\frac{2}{3}-1.57\alpha$	$-3\alpha/2$

Table 2. The cumulants and exponents for the random fixed points for $d=3$ and several values of α. The row labelled W is obtained from the cumulants of W rather than K. The row labelled MC is obtained from the Monte Carlo calculation of section 4.

	c_1	c_2	c_3	c_4	ν	ϕ
$\alpha=0.002$	−0.0172	0.0700	0.0169	0.0067	0.686	−0.01
$\alpha=0$	−0.0148	0.0595	0.0122	0.0041	0.6833	−0.014
$\alpha=0$ W	—	—	—	—	0.6815	−0.066
$\alpha=0, b=2$ MC	−0.007	0.03	—	—	0.685 ± 0.005	—
$\alpha=-0.002$	−0.0110	0.0436	0.0066	0.0016	0.679	−0.004

correlation length exponent satisfies the bound, $\nu > 2/d$. Since the first two cumulants are reasonably small and the third and fourth cumulants are very small one can be hopeful that the low order truncation of the recursion relations gives a good estimate of properties of this fixed point though in the absence of a consistent ordering scheme this cannot be guaranteed.

As a test of the accuracy of calculation we carried out the same analysis using the cumulants of W rather than K, starting from (2.13). The results are shown in table 2 in the row labelled 'W'. The difference between the results of the two truncation schemes gives an estimate of the uncertainty in the value of the exponents.

The non-perturbative fixed point is the only stable fixed point for positive values of α and thus controls the critical behaviour of the system for any strength of disorder. When α is slightly negative, there are three physical fixed points; the pure fixed point, the perturbative fixed point and the non-perturbative fixed point. The perturbative fixed point has two unstable directions and represents a multicritical point separating a line of pure transitions governed by the pure fixed point from a line of random transitions governed by the non-perturbative fixed point.

As α is decreased further, the perturbative and non-perturbative fixed points move together and annihilate one another when $\alpha = \alpha_c \approx -0.003$. Mathematically the perturbative and non-perturbative fixed points continue to exist for $\alpha < \alpha_c$ as unphysical complex conjugate pairs in the space of cumulants. The renormalization group flows in the c_1-c_2 plane are sketched in figure 2 for the several ranges of α values.

3.4. Low temperature phase

There is a connection between the low temperature phase of the Potts model on a hierarchical lattice and the directed self-avoiding walk on the corresponding dual lattice. To see this, note that when K is large, $w(K) \to q \exp(-K)$ so that the recursion relation, (2.4), becomes

$$\tilde{Z} = \sum_{j=1}^{b} \left(\prod_{i=1}^{b^{d-1}} Z_{ij} \right) \tag{3.8}$$

where $Z_{ij} = \exp(-K_{ij})$. Equation (3.8) is the recursion relation for the partition function of a directed walk on a diamond lattice with random bond energies given by the random couplings of the Potts models [9]. The dimensionality, d', of the diamond lattice which is dual to the necklace lattice of dimension d is given by $d' = d/(d-1)$.

If the variance of K is sufficiently large, then the sum in (3.8) can be replaced by its largest term yielding strong disorder recursion relations,

$$\tilde{K} = \min_{j=1}^{b} \left(\sum_{i=1}^{b^{d-1}} K_{ij} \right). \tag{3.9}$$

These are the 'zero temperature' recursion relations for the directed walk [10]. In [9] it is shown that for $d' \leqslant 2$, the flow associated with (3.8) is always towards strong disorder if the initial variance in K is non-vanishing. Thus, for $d \geqslant 2$, (3.9) is the proper recursion relation to investigate the low temperature phase of the random Potts model. The scaling properties of the strong disorder phase defined by (3.9) were studied in [10]. The exponent, ω, characterizes the scaling of the variance of K, $\mathrm{var}(K) \sim L^{2\omega}$ where L is the length scale. For $b = 2$ and $d = 2$ the result [10] is $\omega = 0.3$ which is close to the exact value of $\frac{1}{3}$ for directed walks on Euclidean lattices.

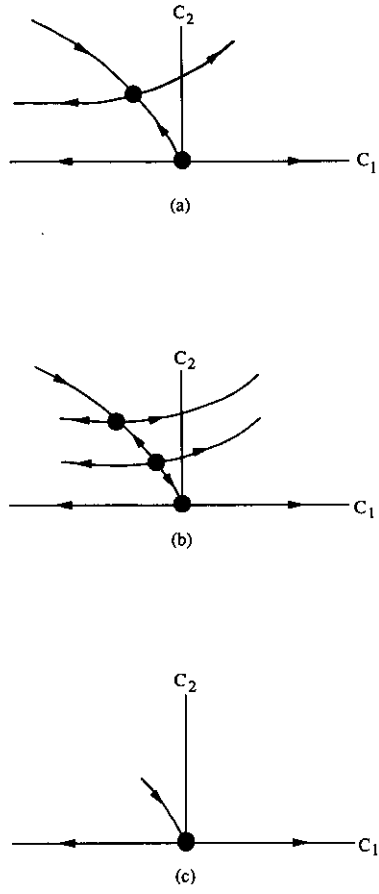

Figure 2. The renormalization group flows for $d=3$ in the c_1-c_2 subspace for (a) $\alpha>0$ (b) $\alpha_c<\alpha<0$ and (c) $\alpha<\alpha_c$.

It is interesting that the low temperature phase of the random Potts model for $d \geqslant 2$ is always a *disordered* phase even though the critical phase may be pure or disordered depending on α and the strength of the disorder. The renormalized coupling, K is related to an intensive quantity, the spin stiffness, Ω, via $\Omega = K/L^{d-1}$. Thus fluctuations in Ω decrease as $L^{\omega+1-d}$. Since pure systems and the systems with infinitesimal disorder flow to different low temperature fixed points, the free energy is non-analytic in the disorder at zero disorder.

4. Numerical results

The multicritical scenario presented in section 3.3 for $d=3$ is quite surprising and one wonders whether it is either an artefact of truncating the recursion relations for the cumulants at finite-order or taking the $b \to 1$ limit. In this section we investigate the $d=3$, $b=2$ recursion relations numerically and confirm the qualitative conclusions of the previous section.

We calculated the fixed distributions and the leading eigenvalues for the Migdal-Kadanoff renormalization group for $b=2$ and $d=3$. In contrast to the histogram

approach of [3] we used a Monte Carlo approach [11] in which the distribution is represented by N double precision numbers where N is taken as 30 000. We used the recursion relations for K (2.4) setting $b=2$ to transform an ensemble of N values of K into a new ensemble of N values of $\tilde{K}$.

We used a version of Newton's method to locate the fixed distribution. If the mean value of the distribution changes under renormalization then a constant times the change is subtracted from each member of the ensemble, i.e. the distribution is shifted in the direction opposite to the change. If the constant is judiciously chosen (1.4 works well) then the process converges to a fixed distribution and the shifts at each stage become quite small (10^{-3} to 10^{-5}).

The qualitative features which we find are in agreement with the analytic results. If $\alpha > 0$ then the fixed distribution has a finite variance and as α approaches zero, this variance does not approach zero. For α slightly less than zero the fixed point depends on the initial variance of K; a delta distribution results if the initial disorder is weak and a broad distribution results if the initial disorder is strong. Finally, for α less than roughly -0.003 only the pure fixed point exists. The fixed distribution for $\alpha = 0$ ($q = 20.896$) is shown in figure 3. The fixed point for the pure system is at $k^* = 0.3827$ and the first two cumulants of this distribution are given in table 2.

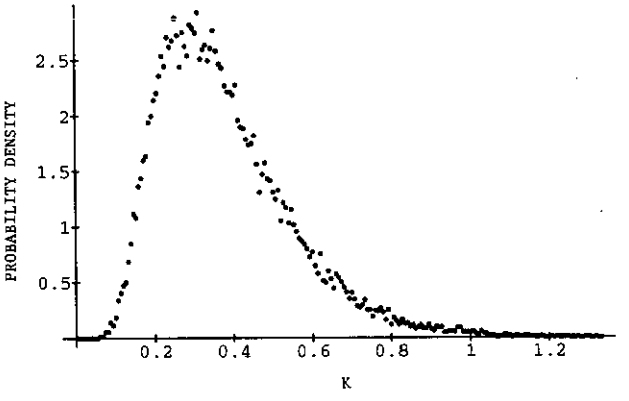

Figure 3. The fixed distribution for $d=3$, $b=2$ and $\alpha=0$.

In order to compute the thermal eigenvalue, we first find the fixed distribution by the above procedure. Then a second copy of the ensemble of random couplings is made and shifted by a small amount, $\delta_0 = 10^{-6}$. The two copies of the distribution are then simultaneously transformed according to the above procedure and the difference, δ_n, between the mean values of the two distributions is recorded at each successive iteration, n. The exponent ν is given by

$$\nu = \frac{\ln(2)}{\ln(\delta_{n+1}/\delta_n)}. \tag{4.1}$$

It is important to note that the small shifts needed to stabilize the original distribution near the fixed point are applied to *both* copies of the distribution. This procedure reduces errors due to the finite size of the Monte Carlo sample. After one or two iterations, the value of ν reaches a plateau which is practically independent of n until the copy of the distribution drifts out of the linear regime after roughly 10 iterations.

The correlation length exponent is shown for $\alpha = 0$ in table 2. The cumulants and exponent obtained by the numerical method are in reasonable agreement with the analytic methods.

The assumption underlying the method for finding the fixed point and its thermal eigenvalue is that the unstable direction in the space of distributions is mainly associated with shifting the distribution. This assumption is supported in part by analytically determining the eigenfunction of the thermal eigenvalue. In all cases studied here, this eigenfunction is dominated by c_1 so that the unstable direction is predominantly a shift in the distribution.

5. Discussion

We have introduced a new method for studying phase transitions in disordered systems. This approach is based upon the Migdal-Kadanoff approximate real space renormalization group. For disordered systems this renormalization group acts on the space of probability distributions for the coupling strengths. We derive an exact flow equation for the Laplace transform of the distribution of coupling strengths. Upon taking the limit $b \to 1$ a differential equation is obtained which serves as a convenient starting point for further calculations. In the present paper we have illustrated the utility of this approach by studying the disordered ferromagnetic Potts models. The same approach should prove to be useful for other phase transitions defined on hierarchical lattices. For example, it is not difficult to obtain analogous differential recursion relations for the directed walk problem [9] or the Potts model on fractally diluted lattices [11].

The most striking result of our investigation of the disordered Potts model is the discovery of a non-perturbative fixed point in three dimensions. When the specific heat exponent is positive this fixed point controls the properties of the disordered system. As α approach zero, the correlation length exponents remain greater than the pure system value of $\frac{2}{3}$. When α is slightly negative, there are three fixed points. The pure fixed point is stable and controls the critical behaviour for weak disorder. The non-perturbative fixed point is also stable and controls the critical behaviour for strong disorder. The third fixed point is perturbatively near the pure fixed point with α as a small parameter. The perturbative fixed point controls a multicritical point separating a line of pure and random second-order transitions.

A second observation which bears further study is that the pure low temperature phase is always unstable to infinitesimal disorder for $d \geqslant 2$. The random low temperature phase has properties which can be obtained by studying the directed self-avoiding walk on the dual lattice.

It would be very interesting to know whether the above scenario with a multicritical point separating pure behaviour from random behaviour holds for spin systems on Euclidean lattices. For $d = 4 - \varepsilon$ dimensions, field theoretic RG studies [12-14] of n-vector models yield a stable perturbative random fixed point for $\alpha > 0$ and only the pure fixed point for $\alpha < 0$. The situation for the Ising model is complicated [12, 15] by the fact that the random fixed point is $O(\varepsilon^{1/2})$ removed from the pure fixed point rather than $O(\varepsilon)$ perhaps indicating the onset of a non-perturbative fixed point as ε is increased. However, apart from the curious $\varepsilon^{1/2}$ behaviour of the Ising model, there is no evidence for the scenario found here. On the other hand, recent Monte Carlo simulations of the random Ising [16] and XY [17] models in $d = 3$, both of which

have small values of α, have yielded exponents which seem to depend on the strength of the disorder and which differ from the ε expansion predictions. It is difficult to interpret these results since the pure system crossover exponents for both of these systems are small and finite-size effects may be important well beyond the length scales available to the simulations.

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Appendix. Properties of the operator P

The linear operator $\boldsymbol{P}$ is defined by the sequence of operations: (1) inverse Laplace transform, (2) change of variables from x to $w(x)$ (3) Laplace transform and (4) multiplication by -1 or, formally,

$$\boldsymbol{P}f = -\boldsymbol{L}[\boldsymbol{L}^{-1}[f] \circ w]. \tag{A1}$$

The Laplace transform and inverse Laplace transform, defined in (2.5) and (2.9), act upon and yield measures respectively. Since $w(x)$ is its own inverse function, $\boldsymbol{P}$ is its own inverse. Consider the action of $\boldsymbol{P}$ on an exponential, e^{-zk}. The inverse transform yields a step function with discontinuity at k,

$$\boldsymbol{L}^{-1}[e^{-zx}] = \theta(x-k). \tag{A2}$$

The change of variables yield

$$\boldsymbol{L}^{-1}[e^{-zx}] \circ w = \theta(w(x)-k). \tag{A3}$$

Finally, taking the Laplace transform yields

$$\boldsymbol{P}[e^{-zk}] = -\int_0^\infty e^{-zx}\, d\theta(w(x)-k). \tag{A4}$$

Since $w = w^{-1}$ and w is a decreasing function we have $\theta(w(x)-k) = 1-\theta(x-w(k))$ and

$$\boldsymbol{P}[e^{-zk}] = e^{-zw(k)}. \tag{A5}$$

Next consider $\boldsymbol{P}$ acting on functions of the form $z^n e^{-zw(k)}$. Multiplication of a Laplace transform by z is equivalent to differentiation, thus

$$\boldsymbol{L}^{-1}[z^n e^{-zw(k)}] = \theta^{(n)}(x-w(k)) \tag{A6}$$

where the superscript n denotes the nth derivative. Define $p(n, k)$ as

$$p(n, k) \equiv \boldsymbol{P}[z^n e^{-zw(k)}]. \tag{A7}$$

From (A5) and the fact that $w = w^{-1}$ we have

$$p(0, k) = e^{-zk}. \tag{A8}$$

Generally, we have

$$p(n, k) = -\boldsymbol{L}[\theta^{(n)}(w(x)-w(k))] = -\int_0^\infty e^{-zx}\, d\theta^{(n)}(w(x)-w(k)). \tag{A9}$$

Differentiation with respect to k yields

$$\frac{\partial}{\partial k} p(n, k) = w'(k) \int_0^\infty e^{-zx} \, d\theta^{(n+1)}(w(x) - w(k)) \tag{A10}$$

yielding the recursion relation,

$$p(n+1, k) = -\frac{1}{w'(k)} \frac{\partial}{\partial k} p(n, k). \tag{A11}$$

For example,

$$p(1, k) = \frac{z}{w'(k)} e^{-zk}. \tag{A12}$$

References

[1] Jayaprakash C, Riedel E K and Wortis M 1978 *Phys. Rev.* B **18** 2244
[2] Kinzel W and Domany E 1981 *Phys. Rev.* B **23** 3421
[3] Andelman D and Berker A N 1984 *Phys. Rev.* B **29** 2630
[4] Derrida B and Gardner E 1984 *J. Phys. A: Math. Gen.* **17** 3223
[5] Kirkpatrick S 1977 *Phys. Rev.* B **15** 1533
Shapiro B 1986 *Phys. Rev.* B **34** 4394
[6] Bakchich A, Benyoussef A, Biaz T and Laanait L 1988 *Phys. Rev.* B **37** 9443
[7] Derrida B, Dickinson H and Yeomans J 1985 *J. Phys. A: Math. Gen.* **18** L53
[8] Chayes J T, Chayes L, Fisher D S and Spencer T 1986 *Phys. Rev. Lett.* **57** 2999
[9] Cook J and Derrida B 1989 *J. Stat. Phys.* **57** 89
[10] Derrida B and Griffiths R B 1989 *Europhys. Lett.* **8** 111
[11] Machta J 1991 *Phys. Rev. Lett.* **66** 169
[12] Khmel'nitsky D E 1975 *Zh. Eksp. Teor. Fiz.* **68** 1960
[13] Lubensky T C 1975 *Phys. Rev.* B **11** 3573
[14] Grinstein G and Luther A 1976 *Phys. Rev.* B **13** 1329
[15] Jayaprakash C and Katz H J 1977 *Phys. Rev.* B **16** 3987
[16] Heuer H-O 1990 *Phys. Rev.* B **42** 6476
[17] Li Y H and Teitel S 1990 *Phys. Rev.* B **41** 11388